CHAPTER 8

Applications in Fluid Mechanics

8.1 INTRODUCTION

The general topic of fluid mechanics encompasses a wide range of problems of interest in engineering applications. The most basic definition of a *fluid* is to state that a fluid is a material that conforms to the shape of its container. Thus, both liquids and gases are fluids. Alternately, it can be stated that a material which, in itself, cannot support shear stresses is a fluid. The reader familiar with the distortion energy theory of solids will recall that geometric distortion is the result of shear stress while normal stress results in volumetric change. Thus, a fluid readily distorts, since the resistance to shear is very low, and such distortion results in flow.

The physical behavior of fluids and gases is very different. The differences in behavior lead to various subfields in fluid mechanics. In general, liquids exhibit constant density and the study of fluid mechanics of liquids is generally referred to as *incompressible flow.* On the other hand, gases are highly compressible (recall Boyle's law from elementary physics [1]) and temperature dependent. Therefore, fluid mechanics problems involving gases are classified as cases of *compressible flow*.

In addition to considerations of compressibility, the relative degree to which a fluid can withstand some amount of shear leads to another classification of fluid mechanics problems. (Regardless of the definition, all fluids can support *some* shear.) The resistance of a fluid to shear is embodied in the material property known as viscosity. In a very practical sense, viscosity is a measure of the "thickness" of a fluid. Consider the differences encountered in stirring a container of water and a container of molasses. The act of stirring introduces shearing stresses in the fluid. The "thinner," less *viscous,* water is easy to stir; the "thicker," more viscous, molasses is harder to stir. The physical effect is represented by the shear stresses applied to the "stirrer" by the fluid. The concept of viscosity is

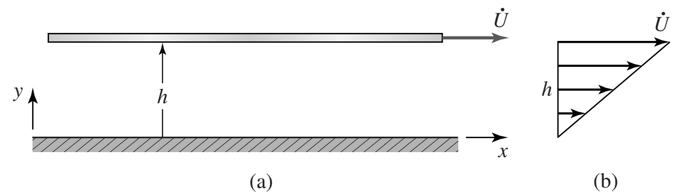


Figure 8.1
(a) Moving plate separated by a fluid layer from a fixed surface. (b) Velocity profile across the fluid thickness.

embodied in Newton's law of viscosity [2], which states that the shear stress in a fluid is proportional to the velocity gradient.

In a one-dimensional case, the velocity gradient and Newton's law of viscosity can be described in reference to Figure 8.1a. A long flat plate is moving with velocity $\dot{U}$ in the x direction and separated from a fixed surface located at $y = 0$ by a thin fluid film of thickness h. Experiments show that the fluid adheres to both surfaces, so that the fluid velocity at the fixed surface is zero, and at the moving plate, the fluid velocity is $\dot{U}$ (this phenomenon is known as the *no slip condition*). If pressure is constant throughout the fluid, the velocity distribution between the moving plate and the fixed surface is linear, as in Figure 8.1b, so the fluid velocity at any point is given by

$$\dot{u}(y) = \frac{y}{h}\dot{U} \tag{8.1}$$

To maintain the motion, a force in the direction of motion must be applied to the plate. The force is required to keep the plate in equilibrium, since the fluid exerts a friction force that opposes the motion. It is known from experiments that the force per unit area (frictional shearing stress) required to maintain motion is proportional to velocity $\dot{U}$ of the moving plate and inversely proportional to distance h. In general, the frictional shearing stress is described in *Newton's law of viscosity* as

$$\tau = \mu \frac{\mathrm{d}\dot{u}}{\mathrm{d}y} \tag{8.2}$$

where the proportionality constant μ is called the *absolute viscosity* of the fluid. Absolute viscosity (hereafter simply *viscosity*) is a fundamental material property of fluid media since, as shown by Equation 8.2, the ability of a fluid to support shearing stress depends directly on viscosity.

The relative importance of viscosity effects leads to yet other subsets of fluid mechanics problems, as mentioned. Fluids that exhibit very little viscosity are termed *inviscid* and shearing stresses are ignored; on the other hand, fluids with significant viscosity must be considered to have associated significant shear effects. To place the discussion in perspective, water is considered to be an incompressible, viscous fluid, whereas air is a highly compressible yet inviscid fluid. In general, liquids are most often treated as incompressible but the viscosity effects depend specifically on the fluid. Gases, on the other hand, are generally treated as compressible but inviscid.

In this chapter, we examine only incompressible fluid flow. The mathematics and previous study required for examination of compressible flow analysis is deemed beyond the scope of this text. We, however, introduce viscosity effects in the context of two-dimensional flow and present the basic finite element formulation for solving such problems. The extension to three-dimensional fluid flow is not necessarily as straightforward as in heat transfer and (as shown in Chapter 9) in solid mechanics. Our introduction to finite element analysis of fluid flow problems shows that the concepts developed thus far in the text can indeed be applied to fluid flow but, in the general case, the resulting equations, although algebraic as expected from the finite element method, are nonlinear and special solution procedures must be applied.

8.2 GOVERNING EQUATIONS FOR INCOMPRESSIBLE FLOW

One of the most important physical laws governing motion of any continuous medium is the principle of conservation of mass. The equation derived by application of this principle is known as the *continuity equation*. Figure 8.2 shows a differential volume (a control volume) located at an arbitrary, fixed position in a three-dimensional fluid flow. With respect to a fixed set of Cartesian axes, the velocity components parallel to the x, y, and z axes are denoted u, v, and w, respectively. (Note that here we take the standard convention of fluid mechanics by denoting velocities without the "dot" notation.) The principle of conservation of mass requires that the time rate of change of mass within the volume must

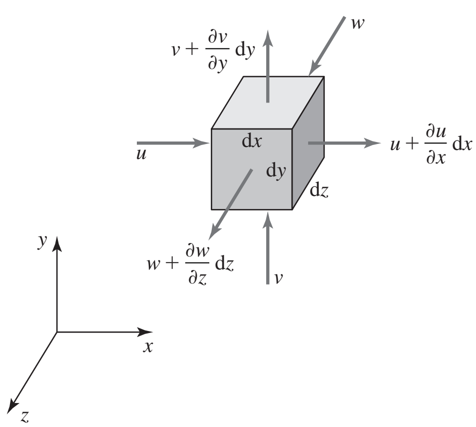


Figure 8.2 Differential volume element in three-dimensional flow.

be in balance with the net mass flow rate into the volume. Total mass inside the volume is $\rho\,\mathrm{d}V$, and since $\mathrm{d}V$ is constant, we must have

$$\frac{\partial\rho}{\partial t}\,\mathrm{d}V = \sum(\text{mass flow in} - \text{mass flow out})$$

and the partial derivative is used because density may vary in space as well as time. Using the velocity components shown, the rate of change of mass in the control volume resulting from flow in the x direction is

$$\dot{m}_x = \rho u\,\mathrm{d}y\,\mathrm{d}z - \left[\rho u + \frac{\partial(\rho u)}{\partial x}\,\mathrm{d}x\right]\mathrm{d}y\,\mathrm{d}z \tag{8.3a}$$

while the corresponding terms resulting from flow in the y and z directions are

$$\dot{m}_y = \rho v\,\mathrm{d}x\,\mathrm{d}z - \left[\rho v + \frac{\partial(\rho v)}{\partial y}\,\mathrm{d}y\right]\mathrm{d}x\,\mathrm{d}z \tag{8.3b}$$

$$\dot{m}_z = \rho w\,\mathrm{d}x\,\mathrm{d}y - \left[\rho w + \frac{\partial(\rho w)}{\partial z}\,\mathrm{d}z\right]\mathrm{d}x\,\mathrm{d}y \tag{8.3c}$$

The rate of change of mass then becomes

$$\frac{\partial\rho}{\partial t}\,\mathrm{d}V = \dot{m}_x + \dot{m}_y + \dot{m}_z = -\left[\frac{\partial(\rho u)}{\partial x} + \frac{\partial(\rho v)}{\partial y} + \frac{\partial(\rho w)}{\partial z}\right]\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z \tag{8.4}$$

Noting that $\mathrm{d}V = \mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z$, Equation 8.4 can be written as

$$\frac{\partial\rho}{\partial t} + u\frac{\partial\rho}{\partial x} + v\frac{\partial\rho}{\partial y} + w\frac{\partial\rho}{\partial z} + \rho\left[\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right] = 0 \tag{8.5}$$

Equation 8.5 is the *continuity equation* for a general three-dimensional flow expressed in Cartesian coordinates.

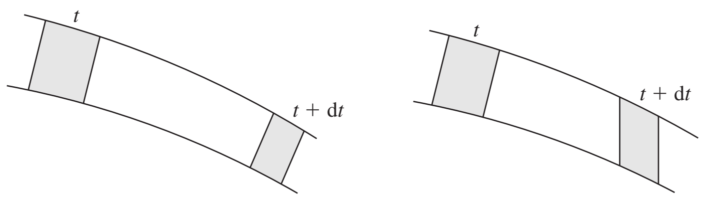
Restricting the discussion to steady flow (with respect to time) of an incompressible fluid, density is independent of time and spatial coordinates so Equation 8.5 becomes

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0 \tag{8.6}$$

Equation 8.6 is the continuity equation for three-dimensional, incompressible, steady flow expressed in Cartesian coordinates. As this is one of the most fundamental equations in fluid flow, we use it extensively in developing the finite element approach to fluid mechanics.

8.2.1 Rotational and Irrotational Flow

Similar to rigid body dynamics, consideration must be given in fluid dynamics as to whether the flow motion represents translation, rotation, or a combination of the two types of motion. Generally, in fluid mechanics, *pure rotation* (i.e.,



(a) (b)

Figure 8.3 Fluid element in (a) rotational flow and (b) irrotational flow.

rotation about a fixed point) is not of as much concern as in rigid body dynamics. Instead, we classify fluid motion as *rotational* (translation and rotation combined) or *irrotational* (translation only). Owing to the inherent deformability of fluids, the definitions of *translation* and *rotation* are not quite the same as for rigid bodies. To understand the difference, we focus on the definition of *rotation* in regard to fluid flow.

A flow field is said to be *irrotational* if a typical element of the moving fluid undergoes no *net* rotation. A classic example often used to explain the concept is that of the passenger carriages on a Ferris wheel. As the wheel turns through one revolution, the carriages also move through a circular path but remain in fixed orientation relative to the gravitational field (assuming the passengers are well-behaved). As the carriage returns to the starting point, the angular orientation of the carriage is exactly the same as in the initial orientation, hence no *net* rotation occurred. To relate the concept to fluid flow, we consider Figure 8.3, depicting two-dimensional flow through a conduit. Figure 8.3a shows an element of fluid undergoing rotational flow. Note that, in this instance, we depict the fluid element as behaving essentially as a solid. The fluid has clearly undergone translation and rotation. Figure 8.3b depicts the same situation in the case of irrotational flow. The element has deformed (angularly), and we indicate that angular deformation via the two angles depicted. If the sum of these two angles is zero, the flow is defined to be irrotational. As is shown in most basic fluid mechanics textbooks [2], the conditions for irrotationality in three-dimensional flow are

$$
\begin{aligned}
\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} &= 0 \\
\frac{\partial w}{\partial x} - \frac{\partial u}{\partial z} &= 0 \\
\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z} &= 0
\end{aligned}
\tag{8.7}
$$

When the expressions given by Equations 8.7 are not satisfied, the flow is rotational and the rotational rates can be defined in terms of the partial derivatives of the same equation. In this text, we consider only irrotational flows and do not proceed beyond the relations of Equation 8.7.

8.3 THE STREAM FUNCTION IN TWO-DIMENSIONAL FLOW

We next consider the case of two-dimensional, steady, incompressible, irrotational flow. (Note that we implicitly assume that viscosity effects are negligible.) Applying these restrictions, the continuity equation is

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \tag{8.8}$$

and the irrotationality conditions reduce to

$$\frac{\partial u}{\partial y} - \frac{\partial v}{\partial x} = 0 \tag{8.9}$$

Equations 8.8 and 8.9 are satisfied if we introduce (define) the *stream function* $\psi(x, y)$ such that the velocity components are given by

$$\begin{aligned} u &= \frac{\partial \psi}{\partial y} \\ v &= -\frac{\partial \psi}{\partial x} \end{aligned} \tag{8.10}$$

These velocity components automatically satisfy the continuity equation. The irrotationality condition, Equation 8.10, becomes

$$\frac{\partial u}{\partial y} - \frac{\partial v}{\partial x} = \frac{\partial}{\partial y}\left(\frac{\partial \psi}{\partial y}\right) - \frac{\partial}{\partial x}\left(-\frac{\partial \psi}{\partial x}\right) = \frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} = \nabla^2 \psi = 0 \tag{8.11}$$

Equation 8.11 is *Laplace's equation* and occurs in the governing equations for many physical phenomena. The symbol ∇ represents the *vector* derivative operator defined, in general, by

$$\nabla = \frac{\partial}{\partial x}\mathbf{i} + \frac{\partial}{\partial y}\mathbf{j} + \frac{\partial}{\partial z}\mathbf{k} \text{ in Cartesian coordinates} \quad \text{and} \quad \nabla^2 = \nabla \cdot \nabla$$

Let us now examine the physical significance of the stream function $\psi(x, y)$ in relation to the two-dimensional flow. In particular, we consider lines in the (x, y) plane (known as *streamlines*) along which the stream function is constant. If the stream function is constant, we can write

$$\mathrm{d}\psi = \frac{\partial \psi}{\partial x}\,\mathrm{d}x + \frac{\partial \psi}{\partial y}\,\mathrm{d}y = 0 \tag{8.12}$$

or

$$\mathrm{d}\psi = -v\,\mathrm{d}x + u\,\mathrm{d}y = 0 \tag{8.13}$$

The tangent vector at any point on a streamline can be expressed as $\mathbf{n}_t = \mathrm{d}x\mathbf{i} + \mathrm{d}y\mathbf{j}$ and the fluid velocity vector at the same point is $\mathbf{V} = u\mathbf{i} + v\mathbf{j}$. Hence, the vector product $\mathbf{V} \times \mathbf{n}_t = (-v\,\mathrm{d}x + u\,\mathrm{d}y)\mathbf{k}$ has zero magnitude, per

Equation 8.13. The vector product of two nonzero vectors is zero only if the vectors are parallel. Therefore, at any point on a streamline, the fluid velocity vector is tangent to the streamline.

8.3.1 Finite Element Formulation

Development of finite element characteristics for fluid flow based on the stream function is straightforward, since (1) the stream function $\psi(x, y)$ is a *scalar* function from which the velocity vector components are derived by differentiation and (2) the governing equation is essentially the same as that for two-dimensional heat conduction. To understand the significance of the latter point, reexamine Equation 7.23 and set $\psi = T$, $k_x = k_y = 1$, $Q = 0$, and $h = 0$. The result is the Laplace equation governing the stream function.

The stream function over the domain of interest is discretized into finite elements having M nodes:

$$\psi(x, y) = \sum_{i=1}^{M} N_i(x, y)\psi_i = [N]\{\psi\} \tag{8.14}$$

Using the Galerkin method, the element residual equations are

$$\int\limits_{A^{(e)}} N_i(x, y)\left(\frac{\partial^2\psi}{\partial x^2} + \frac{\partial^2\psi}{\partial y^2}\right) \mathrm{d}x\,\mathrm{d}y = 0 \qquad i = 1, M \tag{8.15}$$

or

$$\int\limits_{A^{(e)}} [N]^T\left(\frac{\partial^2\psi}{\partial x^2} + \frac{\partial^2\psi}{\partial y^2}\right) \mathrm{d}x\,\mathrm{d}y = 0 \tag{8.16}$$

Application of the Green-Gauss theorem gives

$$\int\limits_{S^{(e)}} [N]^T \frac{\partial\psi}{\partial x} n_x\,\mathrm{d}S - \int\limits_{A^{(e)}} \frac{\partial[N]^T}{\partial x}\frac{\partial\psi}{\partial x}\,\mathrm{d}x\,\mathrm{d}y + \int\limits_{S^{(e)}} [N]^T \frac{\partial\psi}{\partial y} n_y\,\mathrm{d}S$$
$$- \int\limits_{A^{(e)}} \frac{\partial[N]^T}{\partial y}\frac{\partial\psi}{\partial y}\,\mathrm{d}x\,\mathrm{d}y = 0 \tag{8.17}$$

where S represents the element boundary and (n_x, n_y) are the components of the outward unit vector normal to the boundary. Using Equations 8.10 and 8.14 results in

$$\int\limits_{A^{(e)}} \left(\frac{\partial[N]^T}{\partial x}\frac{\partial[N]}{\partial x} + \frac{\partial[N]^T}{\partial y}\frac{\partial[N]}{\partial y}\right) \mathrm{d}x\,\mathrm{d}y\,\{\psi\} = \int\limits_{S^{(e)}} [N]^T(un_y - vn_x)\,\mathrm{d}S \tag{8.18}$$

and this equation is of the form

$$\left[k^{(e)}\right]\{\psi\} = \left\{f^{(e)}\right\} \tag{8.19}$$

The $M \times M$ element stiffness matrix is

$$\left[k^{(e)}\right] = \int_{A^{(e)}} \left(\frac{\partial [N]^T}{\partial x} \frac{\partial [N]}{\partial x} + \frac{\partial [N]^T}{\partial y} \frac{\partial [N]}{\partial y} \right) \mathrm{d}x\, \mathrm{d}y \tag{8.20}$$

and the nodal forces are represented by the $M \times 1$ column matrix

$$\left\{ f^{(e)} \right\} = \int_{S^{(e)}} [N]^T (u n_y - v n_x)\, \mathrm{d}S \tag{8.21}$$

Since the nodal forces are obtained via integration along element boundaries and the unit normals for adjacent elements are equal and opposite, the forces on interelement boundaries cancel during the assembly process. Consequently, the forces defined by Equation 8.21 need be computed only for element boundaries that lie on global boundaries. This observation is in keeping with similar observations made previously in context of other problem types.

8.3.2 Boundary Conditions

As the governing equation for the stream function is a second-order, partial differential equation in two independent variables, four boundary conditions must be specified and satisfied to obtain the solution to a physical problem. The manner in which the boundary conditions are applied to a finite element model is discussed in relation to Figure 8.4a. The figure depicts a flow field between two parallel plates that form a smoothly converging channel. The plates are assumed sufficiently long in the z direction that the flow can be adequately modeled as two-dimensional. Owing to symmetry, we consider only the upper half of the flow field, as in Figure 8.4b. Section *a-b* is assumed to be far enough from the convergent section that the fluid velocity has an x component only. Since we examine only steady flow, the velocity at *a-b* is $U_{ab} =$ constant. A similar argument applies at section *c-d*, far downstream, and we denote the x-velocity component at that section as $U_{cd} =$ constant. How far upstream or downstream is enough to make these assumptions? The answer is a question of solution convergence. The distances involved should increase until there is no discernible difference in the flow solution. As a rule of thumb, the distances should be 10–15 times the width of the flow channel.

As a result of the symmetry and irrotationality of the flow, there can be no velocity component in the y direction along the line $y = 0$ (i.e., the x axis). The velocity along this line is tangent to the line at all values of x. Given these observations, the x axis is a streamline; hence, $\psi = \psi_1 =$ constant along the axis. Similarly, along the surface of the upper plate, there is no velocity component normal to the plate (imprenetrability), so this too must be a streamline along which $\psi = \psi_2 =$ constant. The values of ψ_1 and ψ_2 are two of the required boundary conditions. Recalling that the velocity components are defined as first partial derivatives of the stream function, the stream function must be known only within a constant. For example, a stream function of the form

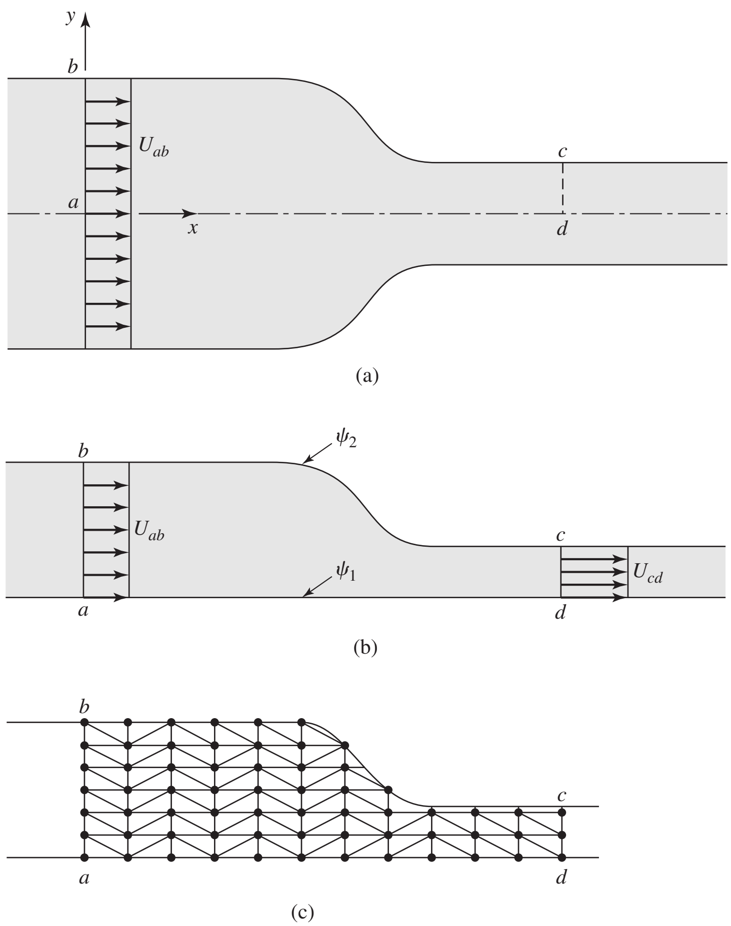


Figure 8.4
(a) Uniform flow into a converging channel. (b) Half-symmetry model showing known velocities and boundary values of the stream function. (c) A relatively coarse finite element model of the flow domain, using three-node triangular elements. This model includes 65 degrees of freedom before applying boundary conditions.

$\psi(x, y) = C + f(x, y)$ contributes no velocity terms associated with the constant C. Hence, one (constant) value of the stream function can be arbitrarily specified. In this case, we choose to set $\psi_1 = 0$. To determine the value of ψ_2, we note that, at section a-b (which we have arbitrarily chosen as $x = 0$, the velocity is

$$u = \frac{\partial \psi}{\partial y} = U_{ab} = \text{constant} = \frac{\psi_2 - \psi_1}{y_b - y_a} = \frac{\psi_2}{y_b} \tag{8.22}$$

so $\psi_2 = y_b U_{ab}$. At any point on *a-b*, we have $\psi = (\psi_2/y_b)y = U_{ab}y$, so the value of the stream function at any finite element node located on *a-b* is known. Similarly, it can be shown that $\psi = (\psi_2/y_c)y = U_{ab}(y_b/y_c)y$ along *c-d*, so nodal values on that line are also known. If these arguments are carefully considered, we see that the boundary conditions on ψ at the "corners" of the domain are continuous and well-defined.

Next we consider the force conditions across sections *a-b* and *c-d*. As noted, the *y*-velocity components along these sections are zero. In addition, the *y* components of the unit vectors normal to these sections are zero as well. Using these observations in conjunction with Equation 8.21, the nodal forces on any element nodes located on these sections are zero. The occurrence of zero forces is equivalent to stating that the streamlines are normal to the boundaries.

If we now utilize a mesh of triangular elements (for example), as in Figure 8.4c, and follow the general assembly procedure, we obtain a set of global equations of the form

$$[K]\{\psi\} = \{F\} \tag{8.23}$$

The forcing function on the right-hand side is zero at all interior nodes. At the boundary nodes on sections *a-b* and *c-d*, we observe that the nodal forces are zero also. At all element nodes situated on the line $y = 0$, the nodal values of the stream function are $\psi = 0$, while at all element nodes on the upper plate profile the values are specified as $\psi = y_b U_{ab}$. The $\psi = 0$ conditions are analogous to the specification of zero displacements in a structural problem. With such conditions, the unknowns are the forces exerted at those nodes. Similarly, the specification of nonzero value of the stream function ψ along the upper plate profile is analogous to a specified displacement. The unknown is the force required to enforce that displacement.

The situation here is a bit complicated mathematically, as we have both zero and nonzero specified values of the nodal variable. In the following, we assume that the system equations have been assembled, and we rearrange the equations such that the column matrix of nodal values is

$$\{\psi\} = \begin{Bmatrix} \{\psi_0\} \\ \{\psi_s\} \\ \{\psi_u\} \end{Bmatrix} \tag{8.24}$$

where $\{\psi_0\}$ represents all nodes along the streamline for which $\psi = 0$, $\{\psi_s\}$ represents all nodes at which the value of ψ is specified, and $\{\psi_u\}$ corresponds to all nodes for which ψ is unknown. The corresponding global force matrix is

$$\{F\} = \begin{Bmatrix} \{F_0\} \\ \{F_s\} \\ \{0\} \end{Bmatrix} \tag{8.25}$$

and we note that all nodes at which ψ is unknown are internal nodes at which the nodal forces are known to be zero.

Using the notation just defined, the system equations can be rewritten (by partitioning the stiffness matrix) as

$$\begin{bmatrix} [K_{00}] & [K_{0s}] & [K_{0u}] \\ [K_{s0}] & [K_{ss}] & [K_{su}] \\ [K_{u0}] & [K_{us}] & [K_{uu}] \end{bmatrix} \begin{Bmatrix} \{\psi_0\} \\ \{\psi_s\} \\ \{\psi_u\} \end{Bmatrix} = \begin{Bmatrix} \{F_0\} \\ \{F_s\} \\ \{0\} \end{Bmatrix} \tag{8.26}$$

Since $\psi_0 = 0$, the first set of partitioned equations become

$$[K_{0s}]\{\psi_s\} + [K_{0u}]\{\psi_u\} = \{F_0\} \tag{8.27}$$

and the values of F_0 can be obtained only after solving for $\{\psi_u\}$ using the remaining equations. Hence, Equation 8.27 is analogous to the reaction force equations in structural problems and can be eliminated from the system temporarily. The remaining equations are

$$\begin{bmatrix} [K_{ss}] & [K_{su}] \\ [K_{us}] & [K_{uu}] \end{bmatrix} \begin{Bmatrix} \{\psi_s\} \\ \{\psi_u\} \end{Bmatrix} = \begin{Bmatrix} \{F_s\} \\ \{0\} \end{Bmatrix} \tag{8.28}$$

and it must be noted that, even though the stiffness matrix is symmetric, $[K_{su}]$ and $[K_{us}]$ are *not* the same. The first partition of Equation 8.28 is also a set of "reaction" equations given by

$$[K_{ss}]\{\psi_s\} + [K_{su}]\{\psi_u\} = \{F_s\} \tag{8.29}$$

and these are used to solve for $\{F_s\}$ but, again, *after* $\{\psi_u\}$ is determined. The second partition of Equation 8.28 is

$$[K_{us}]\{\psi_s\} + [K_{uu}]\{\psi_u\} = \{0\} \tag{8.30}$$

and these equations have the formal solution

$$\{\psi_u\} = -[K_{uu}]^{-1}[K_{us}]\{\psi_s\} \tag{8.31}$$

since the values in $\{\psi_s\}$ are known constants. Given the solution represented by Equation 8.31, the "reactions" in Equations 8.27 and 8.28 can be computed directly.

As the velocity components are of major importance in a fluid flow, we must next utilize the solution for the nodal values of the stream function to compute the velocity components. This computation is easily accomplished given Equation 8.14, in which the stream function is discretized in terms of the nodal values. Once we complete the already described solution procedure for the values of the stream function at the nodes, the velocity components at any point in a specified finite element are

$$\begin{aligned} u(x, y) &= \frac{\partial \psi}{\partial y} = \sum_{i=1}^{M} \frac{\partial N_i}{\partial y}\psi_i = \frac{\partial [N]^T}{\partial y}\{\psi\} \\ v(x, y) &= -\frac{\partial \psi}{\partial x} = -\sum_{i=1}^{M} \frac{\partial N_i}{\partial x}\psi_i = -\frac{\partial [N]^T}{\partial x}\{\psi\} \end{aligned} \tag{8.32}$$

Note that if, for example, a three-node triangular element is used, the velocity components as defined in Equation 8.32 have constant values everywhere in the element and are discontinuous across element boundaries. Therefore, a large number of small elements are required to obtain solution accuracy. Application of the stream function to a numerical example is delayed until we discuss an alternate approach, the velocity potential function, in the next section.

8.4 THE VELOCITY POTENTIAL FUNCTION IN TWO-DIMENSIONAL FLOW

Another approach to solving two-dimensional incompressible, inviscid flow problems is embodied in the velocity potential function. In this method, we hypothesize the existence of a *potential function* $\phi(x, y)$ such that

$$\begin{aligned} u(x, y) &= -\frac{\partial \phi}{\partial x} \\ v(x, y) &= -\frac{\partial \phi}{\partial y} \end{aligned} \tag{8.33}$$

and we note that the velocity components defined by Equation 8.33 automatically satisfy the irrotationality condition. Substitution of the velocity definitions into the continuity equation for two-dimensional flow yields

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = 0 \tag{8.34}$$

and, again, we obtain Laplace's equation as the governing equation for 2-D flow described by a potential function.

We examine the potential formulation in terms of the previous example of a converging flow between two parallel plates. Referring again to Figure 8.4a, we now observe that, along the lines on which the potential function is constant, we can write

$$\mathrm{d}\phi = \frac{\partial \phi}{\partial x}\,\mathrm{d}x + \frac{\partial \phi}{\partial y}\,\mathrm{d}y = -(u\,\mathrm{d}x + v\,\mathrm{d}y) = 0 \tag{8.35}$$

Observing that the quantity $u\,\mathrm{d}x + v\,\mathrm{d}y$ is the magnitude of the scalar product of the velocity vector and the tangent to the line of constant potential, we conclude that the velocity vector at any point on a line of constant potential is *perpendicular* to the line. Hence, the streamlines and lines of constant velocity potential (*equipotential lines*) form an orthogonal "net" (known as the *flow net*) as depicted in Figure 8.5.

The finite element formulation of an incompressible, inviscid, irrotational flow in terms of velocity potential is quite similar to that of the stream function approach, since the governing equation is Laplace's equation in both cases. By

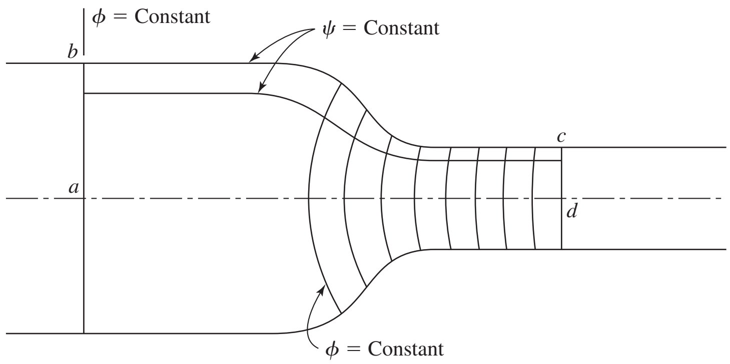


Figure 8.5 Flow net of lines of constant stream function ψ and constant velocity potential ϕ.

direct analogy with Equations 8.14–8.17, we write

$$\phi(x, y) = \sum_{i=1}^{M} N_i(x, y)\phi_i = [N]\{\phi\} \tag{8.36}$$

$$\int_{A^{(e)}} N_i(x, y)\left(\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2}\right) \mathrm{d}x\, \mathrm{d}y = 0 \qquad i = 1, M \tag{8.37}$$

$$\int_{A^{(e)}} \left[N^T\right]\left(\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2}\right) \mathrm{d}x\, \mathrm{d}y = 0 \tag{8.38}$$

$$\int_{S^{(e)}} [N]^T \frac{\partial\phi}{\partial x} n_x\, \mathrm{d}S - \int_{A^{(e)}} \frac{\partial[N]^T}{\partial x}\frac{\partial\phi}{\partial x}\mathrm{d}x\, \mathrm{d}y + \int_{S^{(e)}} [N]^T \frac{\partial\phi}{\partial y} n_y\, \mathrm{d}S - \int_{A^{(e)}} \frac{\partial[N]^T}{\partial y}\frac{\partial\phi}{\partial y}\mathrm{d}x\, \mathrm{d}y = 0 \tag{8.39}$$

Utilizing Equation 8.36 in the area integrals of Equation 8.39 and substituting the velocity components into the boundary integrals, we obtain

$$\int_{A^{(e)}} \left(\frac{\partial[N]^T}{\partial x}\frac{\partial[N]}{\partial x} + \frac{\partial[N]^T}{\partial y}\frac{\partial[N]}{\partial y}\right)\mathrm{d}x\, \mathrm{d}y\, \{\phi\} = -\int_{S^{(e)}} [N]^T (un_x + vn_y)\, \mathrm{d}S \tag{8.40}$$

or

$$\left[k^{(e)}\right]\{\phi\} = \left\{f^{(e)}\right\} \tag{8.41}$$

The element stiffness matrix is observed to be identical to that of the stream function method. The nodal force vector is significantly different, however. Note that, in the right-hand integral in Equation 8.40, the term in parentheses is the scalar product of the velocity vector and the unit normal to an element boundary. Therefore, the nodal forces are allocations to the nodes of the flow across the element boundaries. (Recall that we assume unit dimension in the z direction, so the terms on the right-hand side of Equation 8.40 are volumetric flow rates.) As usual, on internal element boundaries, the contributions from adjacent elements are equal and opposite and cancel during the assembly step. Only elements on global boundaries have nonzero nodal force components.

EXAMPLE 8.1

To illustrate both the stream function and velocity potential methods, we now examine the case of a cylinder placed transversely to an otherwise uniform stream, as shown in Figure 8.6a. The underlying assumptions are

1. Far upstream from the cylinder, the flow field is uniform with $u = U =$ constant and $v = 0$.
2. Dimensions in the z direction are large, so that the flow can be considered two dimensional.
3. Far downstream from the cylinder, the flow is again uniform in accordance with assumption 1.

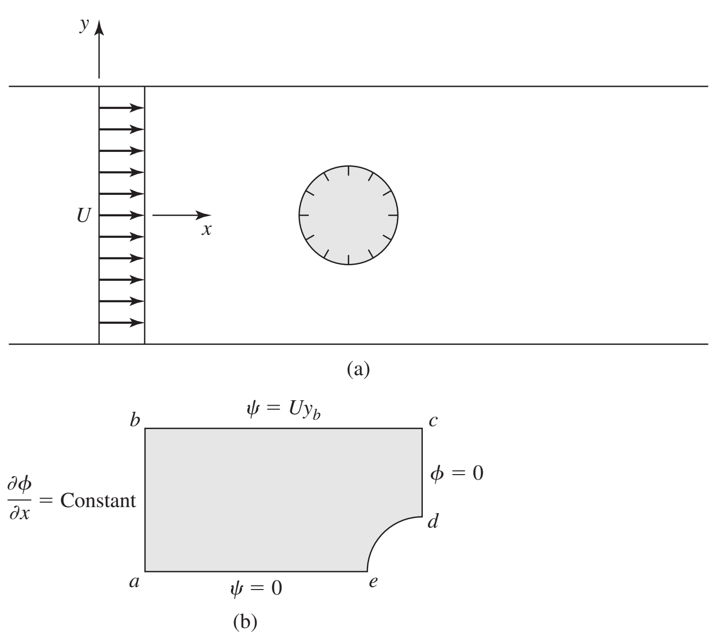


Figure 8.6
(a) Circular cylinder in a uniform, ideal flow. (b) Quarter-symmetry model of cylinder in a uniform stream.

■ Velocity Potential

Given the assumptions and geometry, we need consider only one-fourth of the flow field, as in Figure 8.6b, because of symmetry. The boundary conditions first are stated for the velocity potential formulation. Along $x = 0$ (*a-b*), we have $u = U =$ constant and $v = 0$. So,

$$u(0, y) = U = -\frac{\partial \phi}{\partial x}$$

$$v(0, y) = 0 = -\frac{\partial \phi}{\partial y}$$

and the unit (outward) normal vector to this surface is $(n_x, n_y) = (-1, 0)$. Hence, for every element having edges (therefore, nodes) on *a-b*, the nodal force vector is known as

$$\left\{ f^{(e)} \right\} = -\int_{S^{(e)}} [N]^T (un_x + vn_y)\, \mathrm{d}S = U \int_{S^{(e)}} [N]^T\, \mathrm{d}S$$

and the integration path is simply $\mathrm{d}S = \mathrm{d}y$ between element nodes. Note the change in sign, owing to the orientation of the outward normal vector. Hence, the forces associated with flow into the region are positive and the forces associated with outflow are negative. (The sign associated with inflow and outflow forces depend on the choice of signs in Equation 8.33. If, in Equation 8.33, we choose positive signs, the formulation is essentially the same.)

The symmetry conditions are such that, on surface (edge) *c-d*, the *y*-velocity components are zero and $x = x_c$, so we can write

$$v = -\frac{\partial \phi}{\partial y} = -\frac{\mathrm{d}\phi(x_c, y)}{\mathrm{d}y} = 0$$

This relation can be satisfied if ϕ is independent of the *y* coordinate or $\phi(x_c, y)$ is constant. The first possibility is quite unlikely and requires that we assume the solution form. Hence, the conclusion is that the velocity potential function must take on a constant value on *c-d*. Note, most important, this conclusion *does not* imply that the *x*-velocity component is zero.

Along *b-c*, the fluid velocity has only an *x* component (impenetrability), so we can write this boundary condition as

$$\frac{\partial \phi}{\partial n} = \frac{\partial \phi}{\partial x} n_x + \frac{\partial \phi}{\partial y} n_y = -(un_x + vn_y) = 0$$

and since $v = 0$ and $n_x = 0$ on this edge, we find that all nodal forces are zero along *b-c*, but the values of the potential function are unknown.

The same argument holds for *a-e-d*. Using the symmetry conditions along this surface, there is no velocity perpendicular to the surface, and we arrive at the same conclusion: element nodes have zero nodal force values but unknown values of the potential function.

In summary, for the potential function formulation, the boundary conditions are

1. Boundary *a-b*: ϕ unknown, forces known.
2. Boundary *b-c*: ϕ unknown, forces $= 0$.
3. Boundary *c-d*: $\phi =$ constant, forces unknown.
4. Boundary *a-e-d*: ϕ unknown, forces $= 0$.

Now let us consider assembling the global equations. Per the usual assembly procedure, the equations are of the matrix form

$$[K]\{\phi\} = \{F\}$$

and the force vector on the right-hand side contains both known and unknown values. The vector of nodal potential values $\{\phi\}$ is unknown—we have no specified values. We do know that, along *c-d*, the nodal values of the potential function are constant, but we do not know the value of the constant. However, in light of Equation 8.33, the velocity components are defined in terms of first partial derivatives, so an arbitrary constant in the potential function is of no consequence, as with the stream function formulation. Therefore, we need specify only an arbitrary value of ϕ at nodes on *c-d* in the model, and the system of equations becomes solvable.

■ Stream Function Formulation

Developing the finite element model for this particular problem in terms of the stream function is a bit simpler than for the velocity potential. For reasons that become clear when we write the boundary conditions, we also need consider only one-quarter of the flow field in the stream function approach. The model is also as shown in Figure 8.6b. Along *a-e*, the symmetry conditions are such that the *y*-velocity components are zero. On *e-d*, the velocity components normal to the cylinder must be zero, as the cylinder is impenetrable. Hence, *a-e-d* is a streamline and we arbitrarily set $\psi = 0$ on that streamline. Clearly, the upper surface *b-c* is also a streamline and, using previous arguments from the convergent flow example, we have $\psi = U y_b$ along this edge. (Note that, if we had chosen the value of the stream function along *a-e-d* to be a nonzero value *C*, the value along *b-c* would be $\psi = U y_b + C$.) On *a-b* and *c-d*, the nodal forces are zero, also per the previous discussion, and the nodal values of the stream function are unknown. Except for the geometrical differences, the solution procedure is the same as that for the converging flow.

A relatively coarse mesh of four-node quadrilateral elements used for solving this problem using the stream function is shown in Figure 8.7a. For computation, the values $U = 40$, distance $a\text{-}b = y_b = 5$, and cylinder radius $= 1$ are used. The resulting streamlines (lines of constant ψ) are shown in Figure 8.7b. Recalling that the streamlines are lines to which fluid velocity is tangent at all points, the results appear to be correct intuitively. Note that, on the left boundary, the streamlines appear to be very nearly perpendicular to the boundary, as required if the uniform velocity condition on that boundary is satisfied.

For the problem at hand, we have the luxury of comparing the finite element results with an "approximately exact" solution, which gives the stream function as

$$\psi = U \frac{x^2 + y^2 - R^2}{x^2 + y^2} y$$

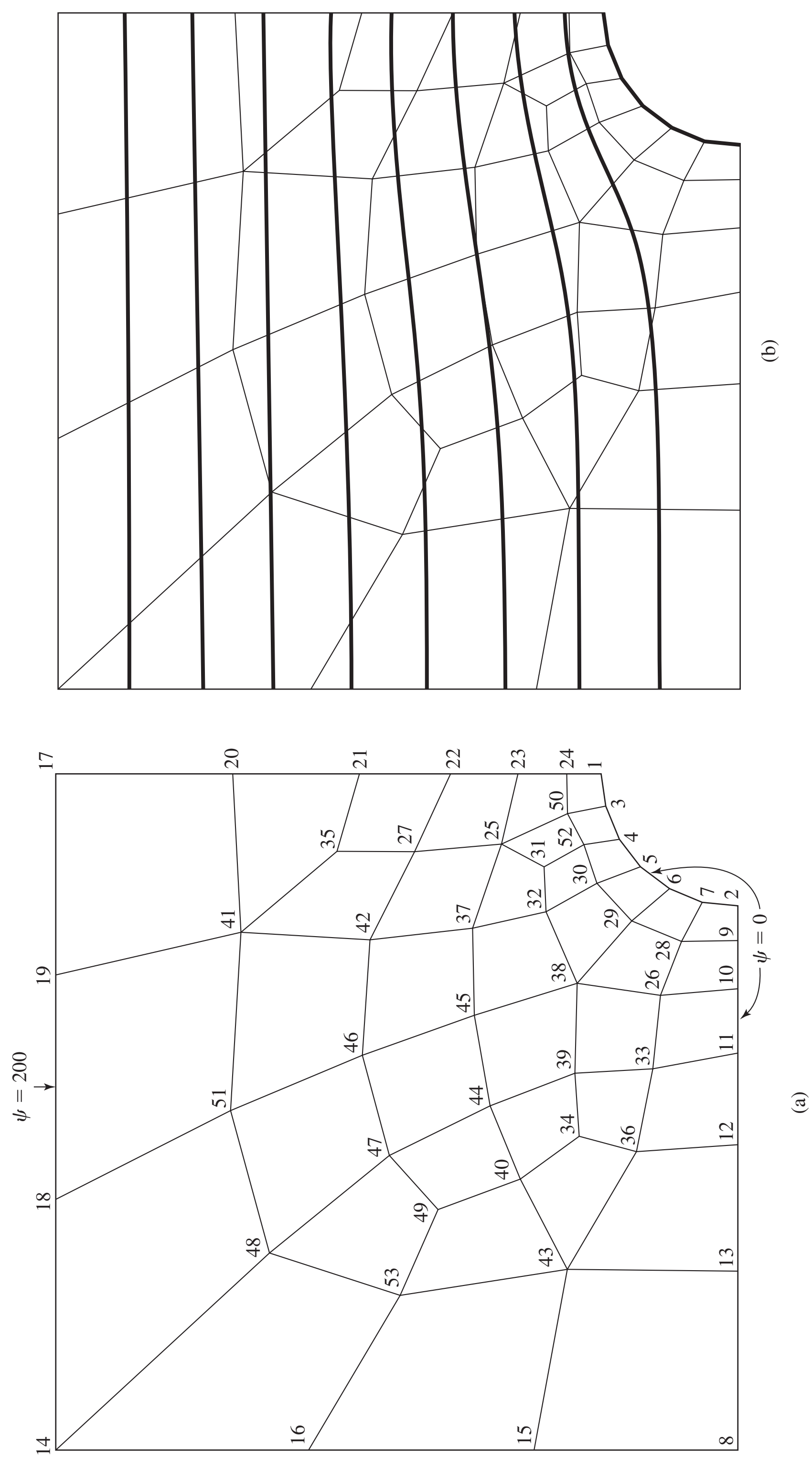


Figure 8.7
(a) Coarse, finite element mesh for stream function solution; 40 elements. (b) Streamlines (ψ = constant) for finite element solution of Example 8.1.

Table 8.1 Selected Nodal Stream Function and Velocity Values for Solution of Example 8.1

Node	ψ_{FE}	ψ_{Exact}	V_{FE}	V_{Exact}
1	0	0	75.184	80
2	0	0	1.963	0
8	0	0	38.735	38.4
16	123.63	122.17	40.533	40.510
20	142.48	137.40	44.903	42.914
21	100.03	99.37	47.109	45.215
22	67.10	64.67	51.535	49.121
23	40.55	39.36	57.836	55.499
24	18.98	18.28	68.142	65.425
45	67.88	65.89	41.706	40.799
46	103.87	100.74	42.359	41.018

This solution is actually for a cylinder in a uniform stream of indefinite extent in both the x and y directions (hence, the use of the oxymoron, approximately exact) but is sufficient for comparison purposes. Table 8.1 lists values of ψ obtained by the finite element solution and the preceding analytical solution at several selected nodes in the model. The computed magnitude of the fluid velocity at those points is also given. The nominal errors in the finite element solution versus the analytical solution are about 4 percent for the value of the stream function and 6 percent for the velocity magnitude. While not shown here, a refined element mesh consisting of 218 elements was used in a second solution and the errors decreased to less than 1 percent for both the stream function value and the velocity magnitude.

Earlier in the chapter, the analogy between the heat conduction problem and the stream function formulation is mentioned. It may be of interest to the reader to note that the stream function solution presented in Example 8.1 is generated using a commercial software package and a two-dimensional heat transfer element. The particular software does not contain a fluid element of the type required for the problem. However, by setting the thermal conductivities to unity and specifying zero internal heat generation, the problem, mathematically, is the same. That is, nodal temperatures become nodal values of the stream function. Similarly, spatial derivatives of temperature (flux values) become velocity components if the appropriate sign changes are taken into account. The mathematical similarity of the two problems is further illustrated by the finite element solution of the previous example using the velocity potential function.

EXAMPLE 8.2

Obtain a finite element solution for the problem of Example 8.1 via the velocity potential approach, using, specifically, the heat conduction formulation modified as required.

■ **Solution**

First let us note the analogies

$$u = -\frac{\partial \phi}{\partial x} \Rightarrow q_x = -k_x \frac{\partial T}{\partial x}$$

$$v = -\frac{\partial \phi}{\partial y} \Rightarrow q_y = -k_y \frac{\partial T}{\partial y}$$

so that, if $k_x = k_y = 1$, then the velocity potential is directly analogous to temperature and the velocity components are analogous to the respective flux terms. Hence, the boundary conditions, in terms of thermal variables become

$$q_x = U \qquad q_y = 0 \quad \text{on } a\text{-}b$$

$$q_x = q_y = 0 \quad \text{on } b\text{-}c \text{ and } a\text{-}e\text{-}d$$

$$T = \text{constant} = 0 \quad \text{on } c\text{-}d \text{ (the value is arbitrary)}$$

Figure 8.8 shows a coarse mesh finite element solution that plots the lines of constant velocity potential ϕ (in the thermal solution, these lines are lines of constant temperature,

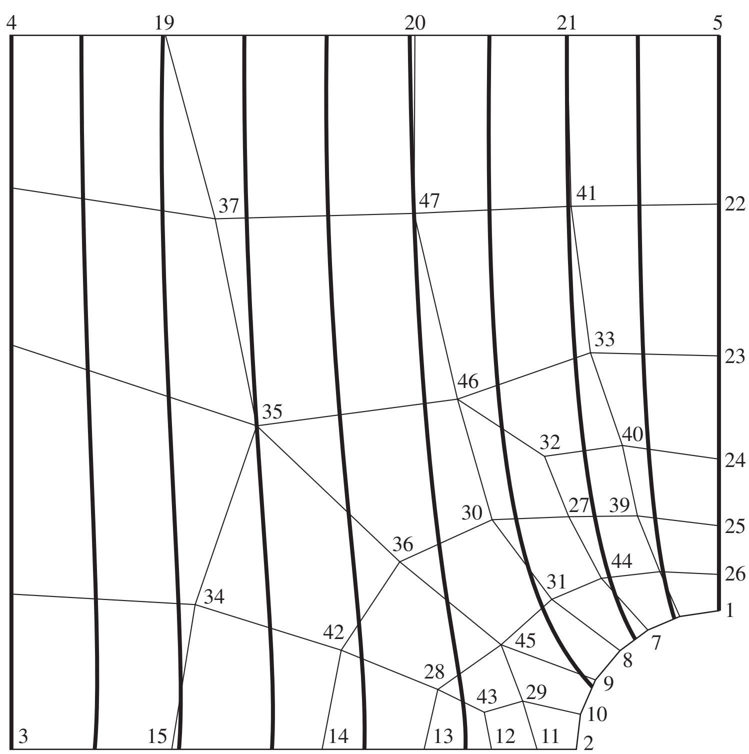


Figure 8.8 Lines of constant velocity potential ϕ for the finite element solution of Example 8.2.

Table 8.2 Velocity Components at Selected Nodes in Example 8.2

Node	u	v
4	40.423	0.480
19	41.019	0.527
20	42.309	0.594
21	43.339	0.516
5	43.676	0.002

or *isotherms*). A direct comparison between this finite element solution and that described for the stream function approach is not possible, since the element meshes are different. However, we can assess accuracy of the velocity potential solution by examination of the results in terms of the boundary conditions. For example, along the upper horizontal boundary, the y-velocity component must be zero, from which it follows that lines of constant ϕ must be perpendicular to the boundary. Visually, this condition appears to be reasonably well-satisfied in Figure 8.8. An examination of the actual data presents a slightly different picture. Table 8.2 lists the computed velocity components at each node along the upper surface. Clearly, the values of the y-velocity component v are not zero, so additional solutions using refined element meshes are in order.

Observing that the stream function and velocity potential methods are amenable to solving the same types of problems, the question arises as to which should be selected in a given instance. In each approach, the stiffness matrix is the same, whereas the nodal forces differ in formulation but require the same basic information. Hence, there is no significant difference in the two procedures. However, if one uses the stream function approach, the flow is readily visualized, since velocity is tangent to streamlines. It can also be shown [2] that the difference in value of two adjacent streamlines is equal to the flow rate (per unit depth) between those streamlines.

8.4.1 Flow around Multiple Bodies

For an ideal (inviscid, incompressible) flow around multiple bodies, the stream function approach is rather straightforward to apply, especially in finite element analysis, if the appropriate boundary conditions can be determined. To begin the illustration, let us reconsider flow around a cylinder as in Example 8.1. Observing that Equation 8.11 governing the stream function is linear, the principle of superposition is applicable; that is, the sum of any two solutions to the equation is also a solution. In particular, we consider the stream function to be given by

$$\psi(x, y) = \psi_1(x, y) + a\psi_2(x, y) \tag{8.42}$$

where a is a constant to be determined. The boundary conditions at the horizontal surfaces (S_1) are satisfied by ψ_1, while the boundary conditions on the surface of the cylinder (S_2) are satisfied by ψ_2. The constant a must be determined so that the combination of the two stream functions satisfies a known condition at some point in the flow. Hence, the conditions on the two solutions (stream functions) are

$$\begin{aligned} \frac{\partial^2 \psi_1}{\partial x^2} + \frac{\partial^2 \psi_1}{\partial y^2} &= 0 \quad \text{(everywhere in the domain)} \\ \frac{\partial^2 \psi_2}{\partial x^2} + \frac{\partial^2 \psi_2}{\partial y^2} &= 0 \quad \text{(everywhere in the domain)} \end{aligned} \tag{8.43}$$

$$\psi_1 = U y_b \quad \text{on } S_1 \tag{8.44}$$

$$\psi_1 = 0 \quad \text{on } S_2 \tag{8.45}$$

$$\psi_2 = 0 \quad \text{on } S_1 \tag{8.46}$$

$$\psi_2 = 1 \quad \text{on } S_2 \tag{8.47}$$

Note that the value of ψ_2 is (temporarily) set equal to unity on the surface of the cylinder. The procedure is then to obtain two finite element solutions, one for each stream function, and associated boundary conditions. Given the two solutions, the constant a can be determined and the complete solution known. The constant a, for example, is found by computing the velocity at a far upstream position (where the velocity is known) and calculating a to meet the known condition.

In the case of uniform flow past a cylinder, the solutions give the trivial result that $a =$ *arbitrary constant,* since we have only one surface in the flow, hence one arbitrary constant. The situation is different if we have multiple bodies, however, as discussed next.

Consider Figure 8.9, depicting two arbitrarily shaped bodies located in an ideal fluid flow, which has a uniform velocity profile at a distance upstream from the two obstacles. In this case, we consider *three* solutions to the governing

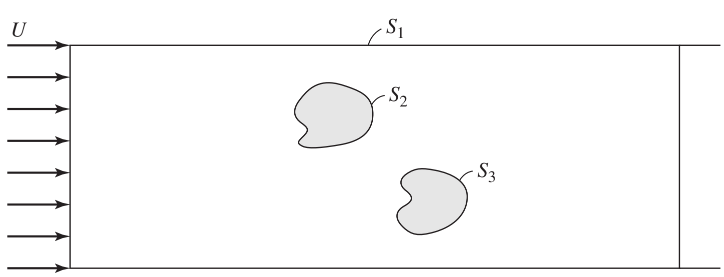


Figure 8.9 Two arbitrary bodies in a uniform stream. The boundary conditions must be specified on S_1, S_2, and S_3 within a constant.

equation, so that the stream function can be represented by [3]

$$\psi(x, y) = \psi_1(x, y) + a\psi_2(x, y) + b\psi_3(x, y) \tag{8.48}$$

where a and b are constants to be determined. Again, we know that each independent solution in Equation 8.48 must satisfy Equation 8.11 and, recalling that the stream function must take on constant value on an impenetrable surface, we can express the boundary conditions on each solution as

$$\begin{aligned} &\psi_1 = U y_b && \text{on } S_1 \\ &\psi_1 = 0 && \text{on } S_2 \text{ and } S_3 \\ &\psi_2 = 0 && \text{on } S_1 \text{ and } S_3 \\ &\psi_2 = 1 && \text{on } S_2 \\ &\psi_3 = 0 && \text{on } S_1 \text{ and } S_2 \\ &\psi_3 = 1 && \text{on } S_3 \end{aligned} \tag{8.49}$$

To obtain a solution for the flow problem depicted in Figure 8.9, we must

1. Obtain a solution for ψ_1 satisfying the governing equation and the boundary conditions stated for ψ_1.
2. Obtain a solution for ψ_2 satisfying the governing equation and the boundary conditions stated for ψ_2.
3. Obtain a solution for ψ_3 satisfying the governing equation and the boundary conditions stated for ψ_3.
4. Combine the results at (in this case) two points, where the velocity or stream function is known in value, to determine the constants a and b in Equation 8.48. For this example, any two points on section *a-b* are appropriate, as we know the velocity is uniform in that section.

As a practical note, this procedure is *not* generally included in finite element software packages. One must, in fact, obtain the three solutions and hand calculate the constants a and b, then adjust the boundary conditions (the constant values of the stream function) for entry into the next run of the software. In this case, not only the computed results (stream function values, velocities) but the values of the computed constants a and b are considerations for convergence of the finite element solutions. The procedure described may seem tedious, and it is to a certain extent, but the alternatives (other than finite element analysis) are much more cumbersome.

8.5 INCOMPRESSIBLE VISCOUS FLOW

The idealized inviscid flows analyzed via the stream function or velocity potential function can reveal valuable information in many cases. Since no fluid is truly inviscid, the accuracy of these analyses decreases with increasing viscosity of a real fluid. To illustrate viscosity effects (and the arising complications) we now examine application of the finite element method to a restricted class of incompressible viscous flows.

The assumptions and restrictions applicable to the following developments are

1. The flow can be considered two dimensional.
2. No heat transfer is involved.
3. Density and viscosity are constant.
4. The flow is steady with respect to time.

Under these conditions, the famed Navier-Stokes equations [4, 5], representing conservation of momentum, can be reduced to [6]

$$\begin{aligned}\rho u\frac{\partial u}{\partial x}+\rho v\frac{\partial u}{\partial y}-\mu\frac{\partial^2 u}{\partial x^2}-\mu\frac{\partial^2 u}{\partial y^2}+\frac{\partial p}{\partial x}&=F_{Bx}\\ \rho u\frac{\partial v}{\partial x}+\rho v\frac{\partial v}{\partial y}-\mu\frac{\partial^2 v}{\partial x^2}-\mu\frac{\partial^2 v}{\partial y^2}+\frac{\partial p}{\partial y}&=F_{By}\end{aligned}\tag{8.50}$$

where

u and v = x-, and y-velocity components, respectively
ρ = density of the fluid
p = pressure
μ = absolute fluid viscosity
F_{Bx}, F_{By} = body force per unit volume in the x and y directions, respectively

Note carefully that Equation 8.50 is nonlinear, owing to the presence of the *convective inertia* terms of the form $\rho u(\partial u/\partial x)$. Rather than treat the nonlinear terms directly at this point, we first consider the following special case.

8.5.1 Stokes Flow

For fluid flow in which the velocities are very small, the inertia terms (i.e., the preceding nonlinear terms) can be shown to be negligible in comparison to the viscous effects. Such flow, known as *Stokes flow* (or *creeping flow*), is commonly encountered in the processing of high-viscosity fluids, such as molten polymers. Neglecting the inertia terms, the momentum equations become

$$\begin{aligned}-\mu\frac{\partial^2 u}{\partial x^2}-\mu\frac{\partial^2 u}{\partial y^2}+\frac{\partial p}{\partial x}&=F_{Bx}\\ -\mu\frac{\partial^2 v}{\partial x^2}-\mu\frac{\partial^2 v}{\partial y^2}+\frac{\partial p}{\partial y}&=F_{By}\end{aligned}\tag{8.51}$$

Equation 8.51 and the continuity condition, Equation 8.8, form a system of three equations in the three unknowns $u(x, y)$, $v(x, y)$, and $p(x, y)$. Hence, a finite element formulation includes three nodal variables, and these are discretized as

$$\begin{aligned} u(x, y) &= \sum_{i=1}^{M} N_i(x, y)u_i = [N]^T\{u\} \\ v(x, y) &= \sum_{i=1}^{M} N_i(x, y)v_i = [N]^T\{v\} \\ p(x, y) &= \sum_{i=1}^{M} N_i(x, y)p_i = [N]^T\{p\} \end{aligned} \tag{8.52}$$

Application of Galerkin's method to a two-dimensional finite element (assumed to have uniform unit thickness in the z direction) yields the residual equations

$$\begin{aligned} &\int_{A^{(e)}} N_i\left(-\mu\frac{\partial^2 u}{\partial x^2} - \mu\frac{\partial^2 u}{\partial y^2} + \frac{\partial p}{\partial x} - F_{Bx}\right)\mathrm{d}A = 0 \\ &\int_{A^{(e)}} N_i\left(-\mu\frac{\partial^2 v}{\partial x^2} - \mu\frac{\partial^2 v}{\partial y^2} + \frac{\partial p}{\partial y} - F_{By}\right)\mathrm{d}A = 0 \qquad i = 1, M \\ &\int_{A^{(e)}} N_i\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right)\mathrm{d}A = 0 \end{aligned} \tag{8.53}$$

As the procedures required to obtain the various element matrices are covered in detail in previous developments, we do not examine Equation 8.53 in its entirety. Instead, only a few representative terms are developed and the remaining results stated by inference.

First, consider the viscous terms containing second spatial derivatives of velocity components such as

$$-\int_{A^{(e)}} \mu N_i\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right)\mathrm{d}A \qquad i = 1, M \tag{8.54}$$

which can be expressed as

$$-\int_{A^{(e)}} \mu\left[\frac{\partial}{\partial x}\left(N_i\frac{\partial u}{\partial x}\right) + \frac{\partial}{\partial y}\left(N_i\frac{\partial u}{\partial y}\right)\right]\mathrm{d}A + \int_{A^{(e)}} \mu\left(\frac{\partial N_i}{\partial x}\frac{\partial u}{\partial x} + \frac{\partial N_i}{\partial y}\frac{\partial u}{\partial y}\right)\mathrm{d}A \qquad i = 1, M \tag{8.55}$$

Application of the Green-Gauss theorem to the first integral in expression (8.55) yields

$$-\int_{A^{(e)}} \mu\left[\frac{\partial}{\partial x}\left(N_i\frac{\partial u}{\partial x}\right)+\frac{\partial}{\partial y}\left(N_i\frac{\partial u}{\partial y}\right)\right]\mathrm{d}A = -\int_{S^{(e)}} \mu N_i\left(\frac{\partial u}{\partial x}n_x+\frac{\partial u}{\partial y}n_y\right)\mathrm{d}S \qquad i=1,M \tag{8.56}$$

where $S^{(e)}$ is the element boundary and (n_x, n_y) are the components of the unit outward normal vector to the boundary. Hence, the integral in expression (8.54) becomes

$$-\int_{A^{(e)}} \mu N_i\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}\right)\mathrm{d}A = -\int_{S^{(e)}} \mu N_i\left(\frac{\partial u}{\partial x}n_x+\frac{\partial u}{\partial y}n_y\right)\mathrm{d}S + \int_{A^{(e)}} \mu\left(\frac{\partial N_i}{\partial x}\frac{\partial u}{\partial x}+\frac{\partial N_i}{\partial y}\frac{\partial u}{\partial y}\right)\mathrm{d}A \tag{8.57}$$

Note that the first term on the right-hand side of Equation 8.57 represents a nodal boundary force term for the element. Such terms arise from shearing stress. As we observed many times, these terms cancel on interelement boundaries and must be considered only on the global boundaries of a finite element model. Hence, these terms are considered only in the assembly step. The second integral in Equation 8.57 is a portion of the "stiffness" matrix for the fluid problem, and as this term is related to the x velocity and the viscosity, we denote this portion of the matrix $[k_{u\mu}]$. Recalling that Equation 8.57 represents M equations, the integral is converted to matrix form using the first of Equation 8.52 to obtain

$$\int_{A^{(e)}} \mu\left(\frac{\partial[N]^T}{\partial x}\frac{\partial[N]}{\partial x}+\frac{\partial[N]^T}{\partial y}\frac{\partial[N]}{\partial y}\right)\mathrm{d}A\,\{u\} = [k_{u\mu}]\{u\} \tag{8.58}$$

Using the same approach with the second of Equation 8.53, the results are similar. We obtain the analogous result

$$-\int_{A^{(e)}} \mu N_i\left(\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}\right)\mathrm{d}A = -\int_{S^{(e)}} \mu N_i\left(\frac{\partial v}{\partial x}n_x+\frac{\partial v}{\partial y}n_y\right)\mathrm{d}S + \int_{A^{(e)}} \mu\left(\frac{\partial N_i}{\partial x}\frac{\partial v}{\partial x}+\frac{\partial N_i}{\partial y}\frac{\partial v}{\partial y}\right)\mathrm{d}A \tag{8.59}$$

Proceeding as before, we can write the area integrals on the right as

$$\int_{A^{(e)}} \mu\left(\frac{\partial[N]^T}{\partial x}\frac{\partial[N]}{\partial x}+\frac{\partial[N]^T}{\partial y}\frac{\partial[N]}{\partial y}\right)\mathrm{d}A\,\{v\} = [k_{v\mu}]\{v\} \tag{8.60}$$

Considering next the pressure terms and converting to matrix notation, the first of Equation 8.53 leads to

$$\int\limits_{A^{(e)}} [N]^T \frac{\partial [N]}{\partial x} \, \mathrm{d}A \, \{p\} = [k_{px}]\{p\} \tag{8.61}$$

and similarly the second momentum equation contains

$$\int\limits_{A^{(e)}} [N]^T \frac{\partial [N]}{\partial y} \, \mathrm{d}A \, \{p\} = [k_{py}]\{p\} \tag{8.62}$$

The nodal force components corresponding to the body forces are readily shown to be given by

$$\begin{aligned} \{f_{Bx}\} &= \int\limits_{A^{(e)}} [N]^T F_{Bx} \, \mathrm{d}A \\ \{f_{By}\} &= \int\limits_{A^{(e)}} [N]^T F_{By} \, \mathrm{d}A \end{aligned} \tag{8.63}$$

Combining the notation developed in Equations 8.58–8.63, the momentum equations for the finite element are

$$\begin{aligned} [k_{u\mu}]\{u\} + [k_{px}]\{p\} &= \{f_{Bx}\} + \{f_{x\tau}\} \\ [k_{v\mu}]\{v\} + [k_{py}]\{p\} &= \{f_{By}\} + \{f_{y\tau}\} \end{aligned} \tag{8.64}$$

where, for completeness, the nodal forces corresponding to the integrals over element boundaries $S^{(e)}$ in Equations 8.57 and 8.59 have been included.

Finally, the continuity equation is expressed in terms of the nodal velocities in matrix form as

$$\int\limits_{A^{(e)}} [N]^T \frac{\partial [N]}{\partial x} \, \mathrm{d}A\{u\} + \int\limits_{A^{(e)}} [N]^T \frac{\partial [N]}{\partial y} \, \mathrm{d}A\{v\} = [k_u]\{u\} + [k_v]\{v\} = 0 \tag{8.65}$$

where

$$\begin{aligned} [k_u] = [k_{px}] &= \int\limits_{A^{(e)}} [N]^T \frac{\partial [N]}{\partial x} \, \mathrm{d}A \\ [k_v] = [k_{py}] &= \int\limits_{A^{(e)}} [N]^T \frac{\partial [N]}{\partial y} \, \mathrm{d}A \end{aligned} \tag{8.66}$$

As formulated here, Equations 8.64 and 8.65 are a system of $3M$ algebraic equations governing the $3M$ unknown nodal values $\{u\}$, $\{v\}$, $\{p\}$ and can be expressed formally as the system

$$\begin{bmatrix} [k_{u\mu}] & [0] & [k_{px}] \\ [0] & [k_{v\mu}] & [k_{py}] \\ [k_u] & [k_v] & [0] \end{bmatrix} \begin{Bmatrix} \{u\} \\ \{v\} \\ \{p\} \end{Bmatrix} = \begin{Bmatrix} \{f_{Bx}\} \\ \{f_{By}\} \\ \{0\} \end{Bmatrix} \Rightarrow \left[k^{(e)}\right]\left\{\delta^{(e)}\right\} = \left\{f^{(e)}\right\} \qquad (8.67)$$

where $[k^{(e)}]$ represents the complete element stiffness matrix. Note that the element stiffness matrix is composed of nine $M \times M$ submatrices, and although the individual submatrices are symmetric, the stiffness matrix is *not* symmetric.

The development leading to Equation 8.67 is based on evaluation of both the velocity components and pressure at the same number of nodes. This is not necessarily the case for a fluid element. Computational research [7] shows that better accuracy is obtained if the velocity components are evaluated at a larger number of nodes than pressures. In other words, the velocity components are discretized using higher-order interpolation functions than the pressure variable. For example, a six-node quadratic triangular element could be used for velocities, while the pressure variable is interpolated only at the corner nodes, using linear interpolation functions. In such a case, Equation 8.66 does not hold.

The arrangement of the equations and associated definition of the element stiffness matrix in Equation 8.67 is based on ordering the nodal variables as

$$\{\delta\}^T = [u_1 \quad u_2 \quad u_3 \quad v_1 \quad v_2 \quad v_3 \quad p_1 \quad p_2 \quad p_3]$$

(using a three-node element, for example). Such ordering is well-suited to illustrate development of the element equations. However, if the global equations for a multielement model are assembled and the global nodal variables are similarly ordered, that is,

$$\{\Delta\}^T = [U_1 \quad U_2 \cdots V_1 \quad V_2 \cdots P_1 \quad P_2 \cdots P_N]$$

the computational requirements are prohibitively inefficient, because the global stiffness has a large *bandwidth*. On the other hand, if the nodal variables are ordered as

$$\{\Delta\}^T = [U_1 \quad V_1 \quad P_1 \quad U_2 \quad V_2 \quad P_2 \cdots U_N \quad V_N \quad P_N]$$

computational efficiency is greatly improved, as the matrix bandwidth is significantly reduced. For a more detailed discussion of banded matrices and associated computational techniques, see [8].

EXAMPLE 8.3

Consider the flow between the plates of Figure 8.4 to be a viscous, creeping flow and determine the boundary conditions for a finite element model. Assume that the flow is fully developed at sections *a-b* and *c-d* and the constant volume flow rate per unit thickness is Q.

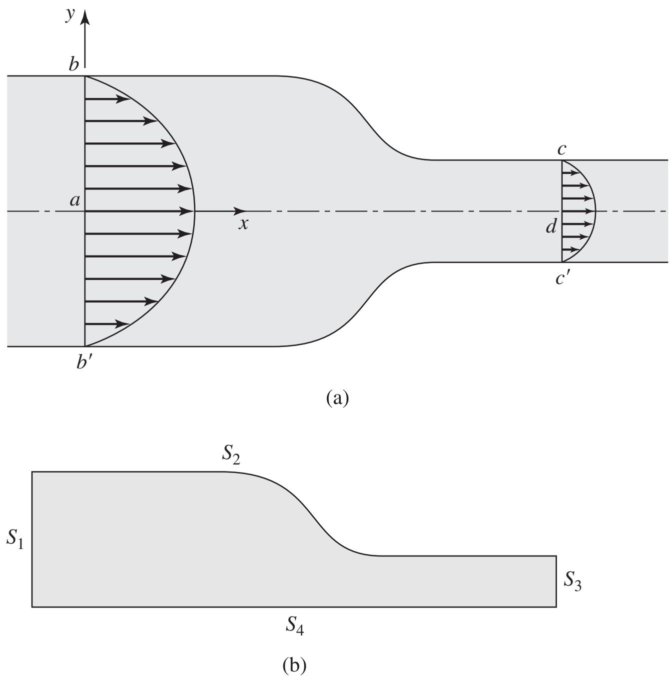


Figure 8.10
(a) Velocity of fully developed flow. (b) Boundary conditions.

■ Solution

For fully developed flow, the velocity profiles at *a-b* and *c-d* are parabolic, as shown in Figure 8.10a. Denoting the maximum velocities at these sections as U_{ab} and U_{cd}, we have

$$u(x_a, y) = U_{ab}\left(1 - \frac{y^2}{y_b^2}\right)$$
$$v(x_a, y) = 0$$
$$u(x_c, y) = U_{cd}\left(1 - \frac{y^2}{y_b^2}\right)$$
$$v(x_c, y) = 0$$

The volume flow rate is obtained by integrating the velocity profiles as

$$Q = 2\int_0^{y_b} u(x_a, y)\,\mathrm{d}y = 2\int_0^{y_d} u(x_c, y)\,\mathrm{d}y$$

Substituting the velocity expressions and integrating yields

$$U_{ab} = \frac{3Q}{4y_b} \qquad U_{cd} = \frac{3Q}{4y_d}$$

and thus the velocity components at all element nodes on *a-b* and *c-d* are known.

Next consider the contact between fluid and plate along *b-c*. As in cases of inviscid flow discussed earlier, we invoke the condition of impenetrability to observe that velocity components normal to this boundary are zero. In addition, since the flow is viscous, we invoke the no-slip condition, which requires that tangential velocity components also be zero at the fluid-solid interface. Hence, for all element nodes on *b-c*, both velocity components u_i and v_i are zero.

The final required boundary conditions are obtained by observing the condition of symmetry along *a-d*, where $v = v(x, 0) = 0$. The boundary conditions are summarized next in reference to Figure 8.10b:

$$U_I = U_{ab}\left(1 - \frac{y_I^2}{y_b^2}\right) \qquad V_I = 0 \quad \text{on } S_1\ (a\text{-}b)$$

$$U_I = U_{cd}\left(1 - \frac{y_I^2}{y_d^2}\right) \qquad V_I = 0 \quad \text{on } S_2\ (b\text{-}c)$$

$$U_I = V_I = 0 \quad \text{on } S_3\ (c\text{-}d)$$

$$V_I = 0 \quad \text{on } S_4\ (a\text{-}d)$$

where I is an element node on one of the global boundary segments.

The system equations corresponding to each of the specified nodal velocities just summarized become constraint equations and are eliminated via the usual procedures prior to solving for the unknown nodal variables. Associated with each specified velocity is an unknown "reaction" force represented by the shear stress-related forces in Equations 8.56, and these forces can be computed using the constraint equations after the global solution is obtained. This is the case for all equations associated with element nodes on segments S_1, S_2, and S_3. On S_4, the situation is a little different and additional comment is warranted. As the velocity components in the x direction along S_4 are not specified, a question arises as to the disposition of the shear-related forces in the x direction. These forces are given by

$$f_{x\tau} = \int\limits_{S^{(e)}} \mu N_i \left(\frac{\partial u}{\partial x} n_x + \frac{\partial u}{\partial y} n_y\right) \mathrm{d}S$$

as embodied in Equation 8.57. On the boundary in question, the unit outward normal vector is defined by $(n_x, n_y) = (0, -1)$, so the first term in this integral is zero. In view of the symmetry conditions about *a-c*, we also have $\partial u/\partial y = 0$, so the shear forces in the x direction along S_4 are also zero. With this observation and the boundary conditions, the global matrix equations become a tractable system of algebraic equations that can be solved for the unknown values of the nodal variables.

8.5.2 Viscous Flow with Inertia

Having discussed slow flows, in which the inertia terms were negligible, we now consider the more general, nonlinear case. All the developments of the previous section on Stokes flow are applicable here; we now add the nonlinear terms arising from the convective inertia terms. From the first of Equation 8.50, we add a term of the form

$$\int_{A^{(e)}} \rho\left(u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y}\right) \mathrm{d}A \Rightarrow \rho \int_{A^{(e)}} \left([N]\{u\}\frac{\partial [N]}{\partial x}\{u\} + [N]\{v\}\frac{\partial [N]}{\partial y}\{u\}\right) \mathrm{d}A \tag{8.68}$$

and from the second equation of 8.50,

$$\int_{A^{(e)}} \rho\left(u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y}\right) \mathrm{d}A \Rightarrow \rho \int_{A^{(e)}} \left([N]\{u\}\frac{\partial [N]}{\partial x}\{v\} + [N]\{v\}\frac{\partial [N]}{\partial y}\{v\}\right) \mathrm{d}A \tag{8.69}$$

As expressed, Equation 8.68 is not conformable to matrix multiplication, as in being able to write the expression in the form $[k]\{u\}$, and this is a direct result of the nonlinearity of the equations. While a complete treatment of the nonlinear equations governing viscous fluid flow is well beyond the scope of this text, we discuss an iterative approximation for the problem.

Let us assume that for a particular two-dimensional geometry, we have solved the Stokes (creeping) flow problem and have all the nodal velocities of the Stokes flow finite element model available. For each element in the finite element model, we denote the Stokes flow solution for the average velocity components (evaluated at the centroid of each element) as $(\bar{u}, \bar{v})$; then, we express the approximation for the inertia terms (as exemplified by Equation 8.69) as

$$\int_{A^{(e)}} \rho\left(u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y}\right) \mathrm{d}A = \int_{A^{(e)}} \rho\left(\bar{u}\frac{\partial [N]}{\partial x} + \bar{v}\frac{\partial [N]}{\partial y}\right) \mathrm{d}A\{u\} = [k_{uv}]\{u\} \tag{8.70}$$

Similarly, we find the y-momentum equation contribution to be

$$\int_{A^{(e)}} \rho\left(u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y}\right) \mathrm{d}A = \int_{A^{(e)}} \rho\left(\bar{u}\frac{\partial [N]}{\partial x} + \bar{v}\frac{\partial [N]}{\partial y}\right) \mathrm{d}A\{v\} = [k_{vu}]\{v\} \tag{8.71}$$

Equations 8.70 and 8.71 refer to an individual element. The assembly procedures are the same as discussed before; now we add additional terms to the stiffness matrix as a result of inertia. These terms are readily identifiable in Equations 8.70 and 8.71. In the viscous inertia flow, the solution requires iteration to achieve satisfactory results. The use of the Stokes flow velocities and pressures represent only the first iteration (approximation). At each iteration, the newly computed velocity components are used for the next iteration.

For both creeping flow and flow with inertia, the governing equations can also be developed in terms of a stream function [3]. However, the resulting (single) governing equation in each case is found to be fourth order. Consequently, elements exhibiting continuity greater than C^0 are required.

8.6 SUMMARY

Application of the finite element method to fluid flow problems is, in one sense, quite straightforward and, in another sense, very complex. In the idealized cases of inviscid flow, the finite element problem is easily formulated in terms of a single variable. Such problems are neither routine nor realistic, as no fluid is truly without viscosity. As shown, introduction of the very real property of fluid viscosity and the historically known, nonlinear governing equations of fluid flow make the finite element method for fluid mechanics analysis difficult and cumbersome, to say the least.

The literature of fluid mechanics is rife with research results on the application of finite element methods to fluid mechanics problems. The literature is so voluminous, in fact, that we do not cite references, but the reader will find that many finite element software packages include fluid elements of various types. These include "pipe elements," "acoustic fluid elements," and "combination elements." The reader is warned to be aware of the restrictions and assumptions underlying the "various sorts" of fluid elements available in a given software package and use care in application.

REFERENCES

1. Halliday, D., R. Resnick, and J. Walker. *Fundamentals of Physics,* 6th ed. New York: John Wiley and Sons, 1997.
2. Crowe, C. T., J. A. Roberson, and D. F. Elger. *Engineering Fluid Mechanics,* 7th ed. New York: John Wiley and Sons, 1998.
3. Huebner, K. H., and E. A. Thornton. *The Finite Element Method for Engineers*. New York: John Wiley and Sons, 1982.
4. Navier, M. "Memoire sur les Lois dur Mouvement des Fluides." *Mem. De l'Acad d. Sci.* (1827).
5. Stokes, G. G. "On the Theories of Internal Friction of Fluids in Motion." *Transactions of the Cambridge Philosophical Society* (1845).
6. Schlichting, H. *Boundary Layer Theory,* 7th ed. New York: McGraw-Hill, 1979.
7. Baker, A. J. *Finite Element Computational Fluid Mechanics.* New York: McGraw-Hill, 1983.
8. Stasa, F. L. *Applied Finite Element Analysis for Engineers.* New York: Holt, Rinehart and Winston, 1985.

PROBLEMS

8.1 Per the standard definition of viscosity described in Section 8.1, how would you describe the property of viscosity, physically, in terms of an everyday example (do not use water and molasses—I already used that example)?

8.2 How would you design an experiment to determine the *relative* viscosity between two fluids? What fluids might you use in this test?

8.3 Look into a fluid mechanics text or reference book. What is the definition of a *Newtonian fluid*?

8.4 Equation 8.5 is a rather complicated partial differential equation, what does it really mean? Explain how that equation takes the very simple form of Equation 8.6.

8.5 If you visually examine a fluid flow, could you determine whether it was rotational or irrotational? Why? Why not?

8.6 Why do we use the Green-Gauss theorem in going from Equation 8.16 to Equation 8.17? Refer to Chapter 5.

8.7 Recalling that Equation 8.21 is based on unit depth in a two-dimensional flow, what do the nodal forces represent physically?

8.8 Given the three-node triangular element shown in Figure P8.8, compute the nodal forces corresponding to the flow conditions shown, assuming unit depth into the plane.

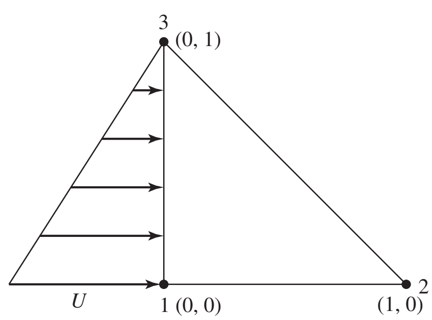


Figure P8.8

8.9 Per Equation 8.32, how do the fluid *velocity* components vary within

a. A linear, three-node triangular element.
b. A four-node rectangular element.
c. A six-node triangular element.
d. An eight-node rectangular element.
e. Given questions *a–d*, how would you decide which element to use in a finite element analysis?

8.10 We show, in this chapter, that both stream function and velocity potential methods are governed by Laplace's equation. Many other physical problems are governed by this equation. Consult mathematical references and find other applications of Laplace's equation. While you are at it (and learning the history of our profession is part of becoming an engineer), find out about Laplace.

8.11 Consider the uniform (ideal) flow shown in Figure P8.11. Use the four triangular elements shown to compute the stream function and derive the velocity components. Note that, in this case, if you do not obtain a uniform flow field, you have made errors in either your formulation or your calculations. The horizontal boundaries are to be taken as fixed surfaces. The coordinates of node 3 are (1.5, 1).

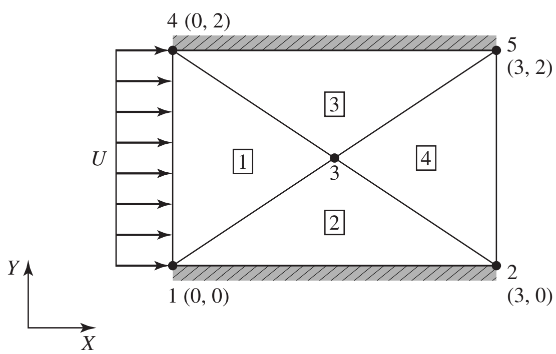


Figure P8.11

8.12 Now repeat Problem 8.11 with the inlet flow shown in Figure P8.12. Does the basic finite element formulation change? Do you have to redefine geometry or elements? Your answer to this question will give you insight as to how to use finite element software. Once the geometry and elements have been defined, various problems can be solved by simply changing boundary conditions or forcing functions.

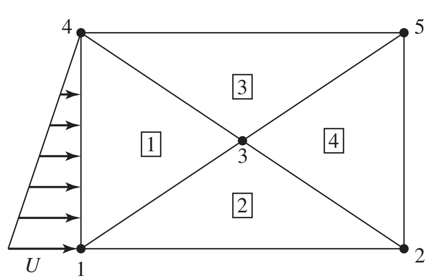


Figure P8.12

8.13 Consider the flow situation depicted in Figure P8.13. Upstream, the flow is uniform. At a known point between the two solid walls, a source of constant strength Q (volume per unit time) exists (via the action of a pump for example). How would the source be accounted for in a finite element formulation? (Examine the heat transfer analogy.)

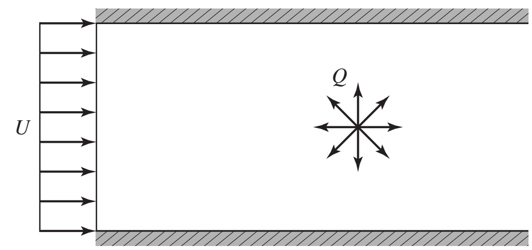


Figure P8.13

8.14 Reconsider Example 8.1 and assume the cylinder is a heating rod held at constant surface temperature T_0. The uniform inlet stream is at known temperature $T_i < T_0$. The horizontal boundaries are perfectly insulated and steady-state conditions are assumed. In the context of finite element analysis, can the flow problem and the heat transfer problem be solved independently?